

# MAPPINGS PRESERVING APPROXIMATE ORTHOGONALITY IN HILBERT $C^*$ -MODULES

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**ABSTRACT.** We introduce a notion of approximate orthogonality preserving mappings between Hilbert  $C^*$ -modules. We define the concept of  $(\delta, \varepsilon)$ -orthogonality preserving mapping and give some sufficient conditions for a linear mapping to be  $(\delta, \varepsilon)$ -orthogonality preserving. In particular, if  $\mathcal{E}$  is a full Hilbert  $\mathcal{A}$ -module with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T, S : \mathcal{E} \rightarrow \mathcal{E}$  are two linear mappings satisfying  $|\langle Sx, Sy \rangle| = \|S\|^2 |\langle x, y \rangle|$  for all  $x, y \in \mathcal{E}$  and  $\|T - S\| \leq \theta \|S\|$ , then we show that  $T$  is a  $(\delta, \varepsilon)$ -orthogonality preserving mapping. We also prove whenever  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T : \mathcal{E} \rightarrow \mathcal{F}$  is a nonzero  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping between  $\mathcal{A}$ -modules, then

$$\|\langle Tx, Ty \rangle - \|T\|^2 \langle x, y \rangle\| \leq \frac{4(\varepsilon - \delta)}{(1 - \delta)(1 + \varepsilon)} \|Tx\| \|Ty\| \quad (x, y \in \mathcal{E}).$$

As a result, we present some characterizations of the orthogonality preserving mappings.

## 1. INTRODUCTION AND PRELIMINARIES

An inner product module over a  $C^*$ -algebra  $\mathcal{A}$  is a (right)  $\mathcal{A}$ -module  $\mathcal{E}$  equipped with an  $\mathcal{A}$ -valued inner product  $\langle \cdot, \cdot \rangle$ , which is  $\mathbb{C}$ -linear and  $\mathcal{A}$ -linear in the second variable and has the properties  $\langle x, y \rangle^* = \langle y, x \rangle$  as well as  $\langle x, x \rangle \geq 0$  with equality if and only if  $x = 0$ . An inner product  $\mathcal{A}$ -module  $\mathcal{E}$  is called a Hilbert  $\mathcal{A}$ -module if it is complete with respect to the norm  $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ . An inner product  $\mathcal{A}$ -module  $\mathcal{E}$  has an “ $\mathcal{A}$ -valued norm”  $|\cdot|$ , defined by  $|x| = \langle x, x \rangle^{\frac{1}{2}}$ . A mapping  $T : \mathcal{E} \rightarrow \mathcal{F}$ , where  $\mathcal{E}$  and  $\mathcal{F}$  are inner product  $\mathcal{A}$ -modules, is called  $\mathcal{A}$ -linear if it is linear and  $T(xa) = (Tx)a$  for all  $x \in \mathcal{E}$ ,  $a \in \mathcal{A}$ .

Although inner product  $C^*$ -modules generalize inner product spaces by allowing inner products to take values in an arbitrary  $C^*$ -algebra instead of the  $C^*$ -algebra of complex numbers, but some fundamental properties of inner

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2010 *Mathematics Subject Classification.* 47B49, 46L05, 46L08.

*Key words and phrases.* Orthogonality preserving mapping, Approximate orthogonality,  $(\delta, \varepsilon)$ -orthogonality preserving mapping, Inner product  $C^*$ -module.

product spaces are no longer valid in inner product  $C^*$ -modules. For example, not each closed submodule of an inner product  $C^*$ -module is complemented. Therefore, when we are studying in inner product  $C^*$ -modules, it is always of some interest to find conditions to obtain the results analogous to those for inner product spaces. We refer the reader to [13] for more information on the basic theory of Hilbert  $C^*$ -modules.

Let  $\mathbb{B}(\mathcal{H})$  be the  $C^*$ -algebra of all bounded operators acting on a complex Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$  and  $\mathbb{K}(\mathcal{H})$  denote the ideal of compact operators. It is well known that the class of Hilbert  $\mathbb{K}(\mathcal{H})$ -modules is a well-behaved class of Hilbert  $C^*$ -modules and they share many nice properties with Hilbert spaces. For example, these structures have orthonormal bases and all closed submodules of such modules are complemented. Many properties of Hilbert  $C^*$ -modules over  $C^*$ -algebras of compact operators can be found in [2].

Given two vectors  $\eta, \zeta$  in a Hilbert space  $\mathcal{H}$ , we shall denote by  $\eta \otimes \zeta \in \mathbb{K}(\mathcal{H})$  the one-rank operator defined by  $(\eta \otimes \zeta)(\xi) = (\xi, \zeta)\eta$ . Obviously,  $\|\eta \otimes \zeta\| = \|\eta\| \|\zeta\|$  and  $\text{tr}(\eta \otimes \zeta) = (\eta, \zeta)$ . Observe that  $\eta \otimes \eta$  is the orthogonal projection to the one dimensional subspace spanned by the unit vector  $\eta$ . If  $T$  is an arbitrary bounded operator on  $\mathcal{H}$ , then  $(\eta \otimes \eta)T(\eta \otimes \eta) = (T\eta, \eta)\eta \otimes \eta$ . This shows that  $\eta \otimes \eta$  is a minimal projection. Recall that a projection (i.e., a self-adjoint idempotent)  $e$  in a  $C^*$ -algebra  $\mathcal{A}$  is called minimal if  $e\mathcal{A}e = \mathbb{C}e$ .

Now let  $\mathcal{E}$  be an inner product (respectively, Hilbert)  $\mathcal{A}$ -module, where  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ . Let  $e = \eta \otimes \eta$  for some unit vector  $\eta \in \mathcal{H}$ , be any minimal projection. Then  $\mathcal{E}_e = \{xe : x \in \mathcal{E}\}$ , is a complex inner product (respectively, Hilbert) space contained in  $\mathcal{E}$  with respect to the inner product  $(x, y) = \text{tr}(\langle x, y \rangle)$ ,  $x, y \in \mathcal{E}_e$ ; see [2]. It is not hard to see that  $\langle x, y \rangle = (x, y)e$ . Note that if  $x \in \mathcal{E}_e$ , then  $\|x\|_{\mathcal{E}_e} = \|x\|_{\mathcal{E}}$ , where the norm  $\|\cdot\|_{\mathcal{E}_e}$  comes from the inner product  $(\cdot, \cdot)$ . This enables us to apply Hilbert space theory by lifting results from the Hilbert space  $\mathcal{E}_e$  to the whole  $\mathcal{A}$ -module  $\mathcal{E}$ .

The orthogonality equation and the related orthogonality preserving property play an important role in Hilbert  $C^*$ -modules, operator algebras,  $K$ -theory and group representation theory; see [1, 3, 8, 11] and the references therein.

Recall that vectors  $\eta, \zeta$  in an inner product  $\mathcal{H}$  are said to be orthogonal, and write  $\eta \perp \zeta$ , if  $(\eta, \zeta) = 0$  and, for a given  $\delta \geq 0$ , they are approximately orthogonal or  $\delta$ -orthogonal, denoted by  $\eta \perp^\delta \zeta$ , if  $|(\eta, \zeta)| \leq \delta \|\eta\| \|\zeta\|$ . For  $\delta \geq 1$ , it is clear that every pair of vectors are  $\delta$ -orthogonal, so the interesting case is when  $\delta \in [0, 1)$ .

A mapping  $T : \mathcal{H} \rightarrow \mathcal{K}$ , where  $\mathcal{H}$  and  $\mathcal{K}$  are inner product spaces, is called orthogonality preserving if  $\eta \perp \zeta \Rightarrow T\eta \perp T\zeta$  ( $\eta, \zeta \in \mathcal{H}$ ). It is known that orthogonality preserving mappings may be nonlinear and discontinuous but under additional assumption of linearity, a mapping  $T$  is orthogonality preserving if and only if it is a scalar multiple of an isometry, that is  $T = \gamma U$ , where  $U$  is an isometry and  $\gamma \geq 0$ ; see [4]. It should be noticed that the same result is obtained in [20] by using a different approach. The orthogonality preserving mappings have been considered also in [15].

Analogously, for  $\delta, \varepsilon \in [0, 1)$ , a mapping  $T : \mathcal{H} \rightarrow \mathcal{K}$  is said to be approximately orthogonality preserving, or  $(\delta, \varepsilon)$ -orthogonality preserving, if  $\eta \perp^\delta \zeta \Rightarrow T\eta \perp^\varepsilon T\zeta$  ( $\eta, \zeta \in \mathcal{H}$ ). Approximately orthogonality preserving mappings have been recently intensively studied in connection with functional analysis and operator theory; cf. [4, 6, 10, 16, 17, 19, 20].

An interesting question is whether a  $(\delta, \varepsilon)$ -orthogonality preserving mapping  $T$  must be close to a linear orthogonality preserving mapping.

In the case where  $\delta = 0$ , Chmieliński [4] and Turnšek [16] verified the properties of mappings that preserve approximate orthogonality in inner product spaces. Also Kong and Cao [10] studied stability of approximate orthogonality preserving mappings and the orthogonality equations. Approximate orthogonality preserving mappings between inner product spaces have been recently considered by Wójcik in [17].

Other approximate orthogonalities in general normed spaces along with the corresponding approximately orthogonality preserving mappings have been studied in [7, 14, 18]. Similar investigations have been carried out in Hilbert spaces in [5, 6, 12].

It is natural to explore the orthogonality preserving mappings between inner product  $C^*$ -modules. So, a mapping  $T : \mathcal{E} \rightarrow \mathcal{F}$ , where  $\mathcal{E}$  and  $\mathcal{F}$  are inner product  $\mathcal{A}$ -modules, is called orthogonality preserving if  $x \perp y \Rightarrow Tx \perp Ty$  ( $x, y \in \mathcal{E}$ ). Also, for  $\delta, \varepsilon \in [0, 1)$ , it is called approximately orthogonality preserving, or  $(\delta, \varepsilon)$ -orthogonality preserving, if

$$\|\langle x, y \rangle\| \leq \delta \|x\| \|y\| \Rightarrow \|\langle Tx, Ty \rangle\| \leq \varepsilon \|Tx\| \|Ty\| \quad (x, y \in \mathcal{E}).$$

The natural problems are to describe such a class of approximately orthogonality preserving mappings and whether each  $(\delta, \varepsilon)$ -orthogonality preserving mapping has to be approximated by an orthogonality preserving mapping.

Ilišević and Turnšek [9] studied approximate orthogonality preserving mappings on  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  in the case when  $\delta = 0$ . Orthogonality preserving mappings have been treated also by Frank et al. [8] and Leung et al. [11].

In this paper, we study  $(\delta, \varepsilon)$ -orthogonality preserving mappings between Hilbert  $\mathcal{A}$ -modules, which generalize some results from [4, 9, 10, 16, 17]. In Section 2, some sufficient conditions for a linear mapping to be  $(\delta, \varepsilon)$ -orthogonality preserving are given. In particular, we show that if  $\mathcal{E}$  is a full Hilbert  $\mathcal{A}$ -module with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T, S : \mathcal{E} \rightarrow \mathcal{E}$  are two linear mappings such that  $|\langle Sx, Sy \rangle| = \|S\|^2 |\langle x, y \rangle|$  for all  $x, y \in \mathcal{E}$  and  $\|T - S\| \leq \theta \|S\|$  with  $\theta \in [0, 1)$ , then  $T$  is a  $(\delta, \varepsilon)$ -orthogonality preserving mapping, where  $\varepsilon = \frac{\theta^2 + 2\theta + \delta}{(1-\theta)^2}$ .

In Section 3 we prove if  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T : \mathcal{E} \rightarrow \mathcal{F}$  is a nonzero  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping between  $\mathcal{A}$ -modules, then

$$\|\langle Tx, Ty \rangle - \|T\|^2 \langle x, y \rangle\| \leq \frac{4(\varepsilon - \delta)}{(1 - \delta)(1 + \varepsilon)} \|Tx\| \|Ty\| \quad (x, y \in \mathcal{E}).$$

As a result, we obtain some characterizations of the orthogonality preserving mappings in inner product  $\mathcal{A}$ -modules. Particularly, we show that a nonzero  $\mathcal{A}$ -linear mapping  $T$  is orthogonality preserving if and only if  $T$  is  $(\varepsilon, \varepsilon)$ -orthogonality preserving. Our results improve some theorems due to Chmieliński [4] and Wójcik [17].

## 2. APPROXIMATE ORTHOGONALITY PRESERVING PROPERTY IN HILBERT $C^*$ -MODULES

In this section, we give some sufficient conditions for a linear mapping to be  $(\delta, \varepsilon)$ -orthogonality preserving. Recall that the minimum modulus  $[T]$  of a linear map  $T$  is defined by  $[T] := \inf\{\|Tx\| : \|x\| = 1\}$ .

**Proposition 2.1.** *Let  $\theta \geq 1$ ,  $\lambda \in [0, \frac{1}{4})$  and  $0 \leq \delta < \frac{1-4\lambda}{\theta^4}$ . Let  $\mathcal{E}$  and  $\mathcal{F}$  be two inner product  $\mathcal{A}$ -modules and let  $T, S : \mathcal{E} \rightarrow \mathcal{F}$  be nonzero linear mappings such that*

- (i)  $\|Tx - Sx\| \leq \lambda \|Sx\|$  for all  $x \in \mathcal{E}$
- (ii)  $\frac{1}{\theta^2} \gamma^2 \|\langle x, y \rangle\| \leq \|\langle Sx, Sy \rangle\| \leq \theta^2 \gamma^2 \|\langle x, y \rangle\|$  for all  $x, y \in \mathcal{E}$ ,

with some  $\gamma \in [[S], \|S\|]$ . Then  $T$  is a  $(\delta, \varepsilon)$ -orthogonality preserving mapping, where  $\varepsilon = \frac{\lambda^2 + 2\lambda + \theta^4 \delta}{(1-\lambda)^2}$ .

*Proof.* It follows from (i) that

$$\|Sx\| = \|Sx - Tx + Tx\| \leq \|Sx - Tx\| + \|Tx\| \leq \lambda\|Sx\| + \|Tx\| \quad (x \in \mathcal{E}).$$

Hence

$$\|Sx\| \leq \frac{1}{1-\lambda}\|Tx\| \quad (x \in \mathcal{E}). \quad (2.1)$$

Put  $y = x$  in (ii) to get  $\|x\| \leq \frac{\theta}{\gamma}\|Sx\|$ , whence by (2.1),

$$\|x\| \leq \frac{\theta}{(1-\lambda)\gamma}\|Tx\| \quad (x \in \mathcal{E}). \quad (2.2)$$

Now, fix  $x, y \in \mathcal{E}$  with  $x \perp^\delta y$ . Hence  $\|\langle x, y \rangle\| \leq \delta\|x\| \|y\|$ . By (i) and (ii), we get

$$\begin{aligned} & \|\langle Tx, Ty \rangle\| \\ & \leq \|\langle Tx, Ty \rangle - \langle Sx, Sy \rangle\| + \|\langle Sx, Sy \rangle\| \\ & \leq \|\langle Tx - Sx, Ty - Sy \rangle + \langle Tx - Sx, Sy \rangle + \langle Sx, Ty - Sy \rangle\| \\ & \quad + \theta^2\gamma^2\|\langle x, y \rangle\| \\ & \leq \|Tx - Sx\| \|Ty - Sy\| + \|Tx - Sx\| \|Sy\| + \|Sx\| \|Ty - Sy\| \\ & \quad + \theta^2\gamma^2\delta\|x\| \|y\| \\ & \leq \lambda^2\|Sx\| \|Sy\| + 2\lambda\|Sx\| \|Sy\| + \theta^2\gamma^2\delta\|x\| \|y\| \quad (\text{by (2.2)}) \\ & \leq (\lambda^2 + 2\lambda)\|Sx\| \|Sy\| + \theta^2\gamma^2\delta \times \frac{\theta^2}{(1-\lambda)^2\gamma^2}\|Tx\| \|Ty\| \quad (\text{by (2.1)}) \\ & \leq (\lambda^2 + 2\lambda) \times \frac{1}{(1-\lambda)^2}\|Tx\| \|Ty\| + \frac{\theta^4\delta}{(1-\lambda)^2}\|Tx\| \|Ty\| \\ & = \frac{\lambda^2 + 2\lambda + \theta^4\delta}{(1-\lambda)^2}\|Tx\| \|Ty\|. \end{aligned}$$

Thus  $\|\langle Tx, Ty \rangle\| \leq \varepsilon\|Tx\| \|Ty\|$  and hence  $Tx \perp^\varepsilon Ty$ .  $\square$

As a consequence, with  $\theta = \sqrt[4]{\frac{\varepsilon}{\delta}}$ ,  $\lambda = 0$  and  $S = T$ , we have

**Corollary 2.2.** *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{E}$  and  $\mathcal{F}$  be two inner product  $\mathcal{A}$ -modules and let  $T : \mathcal{E} \longrightarrow \mathcal{F}$  be a nonzero linear mapping satisfying*

$$\sqrt{\frac{\delta}{\varepsilon}}\gamma^2\|\langle x, y \rangle\| \leq \|\langle Tx, Ty \rangle\| \leq \sqrt{\frac{\varepsilon}{\delta}}\gamma^2\|\langle x, y \rangle\|,$$

*for all  $x, y \in \mathcal{E}$  with some  $\gamma \in [[T], \|T\|]$ . Then  $T$  is a  $(\delta, \varepsilon)$ -orthogonality preserving mapping.*

It follows from the inequality in Corollary 2.2 that  $x \perp y \Rightarrow Tx \perp Ty$  ( $x, y \in \mathcal{E}$ ). In the following we give an example of  $(\delta, \varepsilon)$ -orthogonality preserving mapping between Hilbert  $C^*$ -modules.

**Example 2.3.** Let  $0 < \delta \leq \varepsilon < 1$  and let  $\mathcal{E}$  and  $\mathcal{F}$  be two inner product  $\mathcal{A}$ -modules. We define  $T : \mathcal{E} \rightarrow \mathcal{F}$  by  $Tx = \sqrt{\frac{\varepsilon}{\delta}}x$ . Suppose that  $x, y \in \mathcal{E}$  satisfies  $x \perp^\delta y$ . Hence  $\|\langle x, y \rangle\| \leq \delta \|x\| \|y\|$ . Therefore, we get

$$\begin{aligned} \|\langle Tx, Ty \rangle\| &= \frac{\varepsilon}{\delta} \|\langle x, y \rangle\| \leq \varepsilon \|x\| \|y\| = \delta \left\| \sqrt{\frac{\varepsilon}{\delta}}x \right\| \left\| \sqrt{\frac{\varepsilon}{\delta}}y \right\| \\ &= \delta \|Tx\| \|Ty\| \leq \varepsilon \|Tx\| \|Ty\|. \end{aligned}$$

Thus  $Tx \perp^\varepsilon Ty$ . This shows that  $T$  is a  $(\delta, \varepsilon)$ -orthogonality preserving mapping. In addition, if we consider  $Tx = \sqrt{\frac{\varepsilon}{\delta}}\|x\|x$ , then for all  $x, y \in \mathcal{E}$ , the condition  $x \perp^\delta y$  implies  $Tx \perp^\varepsilon Ty$  but  $T$  is not linear.

For inner product  $\mathcal{A}$ -module  $\mathcal{E}$  we define the relation which is connected with the notion of angle. Fix  $\delta, \varepsilon \in [0, 1)$  and  $c \in \mathcal{A}$  with  $\|c\| < 1$ . Let us say  $\angle_c^\delta$  if  $\left\| \langle x, y \rangle - \|x\| \|y\| c \right\| \leq \delta \|x\| \|y\|$ . A mapping  $T : \mathcal{E} \rightarrow \mathcal{F}$ , where  $\mathcal{E}$  and  $\mathcal{F}$  are inner product  $\mathcal{A}$ -modules, is called  $(\delta, \varepsilon, c)$ -angle preserving, if  $x \angle_c^\delta y \Rightarrow Tx \angle_c^\varepsilon Ty$  ( $x, y \in \mathcal{E}$ ). It is easy to see that  $T$  is a  $(\delta, \varepsilon, 0)$ -angle preserving mapping if and only if  $T$  is  $(\delta, \varepsilon)$ -orthogonality preserving.

**Theorem 2.4.** Let  $\mathcal{E}$  be a full Hilbert  $\mathcal{A}$ -module with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  such that  $\dim \mathcal{H} > 1$  and let a nonzero bounded linear mapping  $S : \mathcal{E} \rightarrow \mathcal{E}$  satisfy

$$|\langle Sx, Sy \rangle| = \|S\|^2 |\langle x, y \rangle| \quad (x, y \in \mathcal{E}). \quad (2.3)$$

Let  $c \in \mathcal{A}$  with  $\|c\| < 1$ ,  $\delta \in [0, 1 - \|c\|)$  and  $\theta \in [0, 1)$ . If a linear mapping  $T : \mathcal{E} \rightarrow \mathcal{E}$  satisfies  $\|T - S\| \leq \theta \|S\|$ , then  $T$  is  $(\delta, \varepsilon, c)$ -angle preserving, where  $\varepsilon = \frac{\theta^2 + 2\theta + \delta + (\theta^2 - 2\theta - 2)\|c\|}{(1 - \theta)^2}$ .

*Proof.* For  $x = z$  and  $y = z$ , (2.3) becomes  $\|Sz\| = \|S\| \|z\|$ . This implies

$$\left| \|Tz\| - \|S\| \|z\| \right| = \left| \|Tz\| - \|Sz\| \right| \leq \|Tz - Sz\| \leq \|T - S\| \|z\| \leq \theta \|S\| \|z\|.$$

Thus

$$\|Tz\| \leq (1 + \theta) \|S\| \|z\| \quad \text{and} \quad \|z\| \leq \frac{\|Tz\|}{\|S\|(1 - \theta)} \quad (z \in \mathcal{E}). \quad (2.4)$$

From (2.3) we have  $\left| \left\langle \frac{S}{\|S\|}x, \frac{S}{\|S\|}y \right\rangle \right| = |\langle x, y \rangle|$  ( $x, y \in \mathcal{E}$ ). So  $\frac{S}{\|S\|}$  preserves the absolute value of the  $\mathcal{A}$ -valued inner product on  $\mathcal{E}$ . By the Wigner's theorem

[3, Theorem 1] there exist an  $\mathcal{A}$ -linear isometry  $U : \mathcal{E} \longrightarrow \mathcal{E}$  and a phase function  $\varphi : \mathcal{E} \longrightarrow \mathbb{C}$  (i.e. its values are of modulus 1) such that

$$\frac{S}{\|S\|}z = \varphi(z)Uz \quad (z \in \mathcal{E}). \quad (2.5)$$

Now, let  $x, y \in \mathcal{E}$  and  $x \angle_c^\delta y$ . By (2.4), we get

$$\|x\| \|y\| - \frac{1}{\|S\|^2} \|Tx\| \|Ty\| \leq \frac{1}{\|S\|^2} \left( \frac{1}{(1-\theta)^2} - 1 \right) \|Tx\| \|Ty\| \quad (2.6)$$

and

$$\frac{1}{\|S\|^2} \|Tx\| \|Ty\| - \|x\| \|y\| \leq \frac{1}{\|S\|^2} \left( 1 - \frac{1}{(1+\theta)^2} \right) \|Tx\| \|Ty\|. \quad (2.7)$$

Since  $1 - \frac{1}{(1+\theta)^2} \leq \frac{1}{(1-\theta)^2} - 1 = \frac{2\theta - \theta^2}{(1-\theta)^2}$ , (2.6) and (2.7) yield

$$\left| \|x\| \|y\| - \frac{1}{\|S\|^2} \|Tx\| \|Ty\| \right| \leq \frac{2\theta - \theta^2}{\|S\|^2 (1-\theta)^2} \|Tx\| \|Ty\|. \quad (2.8)$$

Further, by (2.5) we get

$$\begin{aligned} & \left\| \left\langle \frac{T}{\|S\|}x, \frac{T}{\|S\|}y \right\rangle - \langle \varphi(x)Ux, \varphi(y)Uy \rangle \right\| \\ &= \left\| \left\langle \frac{T}{\|S\|}x, \frac{T}{\|S\|}y \right\rangle - \left\langle \frac{S}{\|S\|}x, \frac{S}{\|S\|}y \right\rangle \right\| \\ &= \frac{1}{\|S\|^2} \left\| \langle Tx - Sx, Ty - Sy \rangle + \langle Tx - Sx, Sy \rangle + \langle Sx, Ty - Sy \rangle \right\| \\ &\leq \frac{1}{\|S\|^2} \left( \|Tx - Sx\| \|Ty - Sy\| + \|Tx - Sx\| \|Sy\| + \|Sx\| \|Ty - Sy\| \right) \\ &\leq \frac{1}{\|S\|^2} \left( \|T - S\|^2 \|x\| \|y\| + \|T - S\| \|S\| \|x\| \|y\| \right. \\ &\quad \left. + \|S\| \|T - S\| \|x\| \|y\| \right) \end{aligned} \quad (2.9)$$

$$\begin{aligned} &\leq \frac{1}{\|S\|^2} \left( \theta^2 \|S\|^2 \|x\| \|y\| + \theta \|S\|^2 \|x\| \|y\| + \theta \|S\|^2 \|x\| \|y\| \right) \\ &= (\theta^2 + 2\theta) \|x\| \|y\| \quad (\text{by (2.4)}) \\ &\leq \frac{\theta^2 + 2\theta}{\|S\|^2 (1-\theta)^2} \|Tx\| \|Ty\|. \end{aligned} \quad (2.10)$$

Since  $x \angle_c^\delta y$ , we have  $\left\| \langle x, y \rangle - \|x\| \|y\| c \right\| \leq \delta \|x\| \|y\|$  and so we obtain

$$\left\| \langle \varphi(x)Ux, \varphi(y)Uy \rangle - \overline{\varphi(x)}\varphi(y)\|x\| \|y\|c \right\| \quad (2.11)$$

$$= |\overline{\varphi(x)}| |\varphi(y)| \left\| \langle U^*Ux, y \rangle - \|x\| \|y\|c \right\|$$

$$= \left\| \langle x, y \rangle - \|x\| \|y\|c \right\|$$

$$\leq \delta \|x\| \|y\| \quad (\text{by (2.4)})$$

$$\leq \frac{\delta}{\|S\|^2(1-\theta)^2} \|Tx\| \|Ty\|. \quad (2.12)$$

From (2.8) it follows that

$$\left\| \overline{\varphi(x)}\varphi(y)\|x\| \|y\|c - \frac{\overline{\varphi(x)}\varphi(y)}{\|S\|^2} \|Tx\| \|Ty\|c \right\| \quad (2.13)$$

$$= |\overline{\varphi(x)}| |\varphi(y)| \left| \|x\| \|y\| - \frac{1}{\|S\|^2} \|Tx\| \|Ty\| \right| \|c\|$$

$$\leq \frac{2\theta - \theta^2}{\|S\|^2(1-\theta)^2} \|Tx\| \|Ty\| \|c\|. \quad (2.14)$$

Also, notice that

$$\left\| \frac{\overline{\varphi(x)}\varphi(y)}{\|S\|^2} \|Tx\| \|Ty\|c - \frac{1}{\|S\|^2} \|Tx\| \|Ty\|c \right\| \quad (2.15)$$

$$= \frac{1}{\|S\|^2} \|Tx\| \|Ty\| |\overline{\varphi(x)}\varphi(y) - 1| \|c\|$$

$$\leq \frac{1}{\|S\|^2} \|Tx\| \|Ty\| \left( |\overline{\varphi(x)}| |\varphi(y)| + 1 \right) \|c\| = \frac{2}{\|S\|^2} \|Tx\| \|Ty\| \|c\|. \quad (2.16)$$



Now, we observe that

$$\begin{aligned}
& \left\| \langle Tx, Ty \rangle - \|Tx\| \|Ty\| c \right\| \\
& \leq \|S\|^2 \left( \left\| \left\langle \frac{T}{\|S\|} x, \frac{T}{\|S\|} y \right\rangle - \langle \varphi(x)Ux, \varphi(y)Uy \rangle \right\| \right. \\
& \quad + \left\| \langle \varphi(x)Ux, \varphi(y)Uy \rangle - \overline{\varphi(x)}\varphi(y)\|x\| \|y\|c \right\| \\
& \quad + \left\| \overline{\varphi(x)}\varphi(y)\|x\| \|y\|c - \frac{\overline{\varphi(x)}\varphi(y)}{\|S\|^2} \|Tx\| \|Ty\|c \right\| \\
& \quad \left. + \left\| \frac{\overline{\varphi(x)}\varphi(y)}{\|S\|^2} \|Tx\| \|Ty\|c - \frac{1}{\|S\|^2} \|Tx\| \|Ty\|c \right\| \right) \\
& \quad \left( \text{by (2.9), (2.11), (2.13) and (2.15)} \right) \\
& \leq \|S\|^2 \left( \frac{\theta^2 + 2\theta}{\|S\|^2(1-\theta)^2} \|Tx\| \|Ty\| + \frac{\delta}{\|S\|^2(1-\theta)^2} \|Tx\| \|Ty\| \right. \\
& \quad \left. + \frac{2\theta - \theta^2}{\|S\|^2(1-\theta)^2} \|Tx\| \|Ty\| \|c\| + \frac{2}{\|S\|^2} \|Tx\| \|Ty\| \|c\| \right) \\
& = \frac{\theta^2 + 2\theta + \delta + (\theta^2 - 2\theta - 2)\|c\|}{(1-\theta)^2} \|Tx\| \|Ty\|.
\end{aligned}$$

Thus  $\left\| \langle Tx, Ty \rangle - \|Tx\| \|Ty\| c \right\| \leq \varepsilon \|Tx\| \|Ty\|$  and hence  $Tx \angle_c^\varepsilon Ty$ .  $\square$

As a consequence, with  $c = 0$ , we have

**Corollary 2.5.** *Let  $\delta, \theta \in [0, 1)$ . Let  $\mathcal{E}$  be a full Hilbert  $\mathcal{A}$ -module with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  such that  $\dim \mathcal{H} > 1$  and let a nonzero bounded linear mapping  $S : \mathcal{E} \rightarrow \mathcal{E}$  satisfying*

$$|\langle Sx, Sy \rangle| = \|S\|^2 |\langle x, y \rangle| \quad (x, y \in \mathcal{E}).$$

*If a linear mapping  $T : \mathcal{E} \rightarrow \mathcal{E}$  satisfies  $\|T - S\| \leq \theta \|S\|$ , then  $T$  is  $(\delta, \varepsilon)$ -orthogonality preserving, where  $\varepsilon = \frac{\theta^2 + 2\theta + \delta}{(1-\theta)^2}$ .*

### 3. MAPPINGS PRESERVING APPROXIMATE ORTHOGONALITY IN HILBERT $C^*$ -MODULES

In this section, we study  $(\delta, \varepsilon)$ -orthogonality preserving mappings between Hilbert  $\mathcal{A}$ -modules whenever  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ . To achieve our main result we prove first some auxiliary results.

**Proposition 3.1.** *Let  $T : \mathcal{H} \longrightarrow \mathcal{K}$  be a  $(\delta, \varepsilon)$ -orthogonality preserving linear mapping. If  $\eta, \zeta \in \mathcal{H}$  are orthogonal unit vectors, then*

$$\sqrt{\frac{(n+1)(1-\delta)(1-\varepsilon)}{n(1+\delta)(1+\varepsilon)}} \|T\zeta\| \leq \|T\eta\| \leq \sqrt{\frac{(n+1)(1-\delta)(1+\varepsilon)}{n(1+\delta)(1-\varepsilon)}} \|T\zeta\|$$

for all  $n \in \mathbb{N}$ .

*Proof.* Let  $n \in \mathbb{N}$ . We have

$$\begin{aligned} & \left| \left( \eta + \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} \zeta, \eta - \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} \zeta \right) \right| \\ &= 1 - \frac{(n+1)(1-\delta)}{n(1+\delta)} \\ &\leq \delta \left[ 1 + \frac{(n+1)(1-\delta)}{n(1+\delta)} \right] \\ &= \delta \left\| \eta + \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} \zeta \right\| \left\| \eta - \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} \zeta \right\|. \end{aligned}$$

So, we get  $\eta + \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} \zeta \perp^\delta \zeta - \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} \zeta$ . Since  $T$  is a  $(\delta, \varepsilon)$ -orthogonality preserving mapping, we reach

$$T\eta + \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta \perp^\varepsilon T\eta - \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta.$$

Therefore,

$$\begin{aligned} & \left| \left( T\eta + \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta, T\eta - \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta \right) \right| \\ &\leq \varepsilon \left\| T\eta + \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta \right\| \left\| T\eta - \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta \right\|, \end{aligned}$$

whence

$$\begin{aligned} & \left( \|T\eta\|^2 - \frac{(n+1)(1-\delta)}{n(1+\delta)} \|T\zeta\|^2 \right)^2 + 4 \left[ \operatorname{Im}(T\eta, \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta) \right]^2 \\ &\leq \varepsilon^2 \left( \left( \|T\eta\|^2 + \frac{(n+1)(1-\delta)}{n(1+\delta)} \|T\zeta\|^2 \right)^2 \right. \\ &\quad \left. - 4 \left[ \operatorname{Re}(T\eta, \sqrt{\frac{(n+1)(1-\delta)}{n(1+\delta)}} T\zeta) \right]^2 \right). \end{aligned}$$

It follows that

$$\left| \|T\eta\|^2 - \frac{(n+1)(1-\delta)}{n(1+\delta)} \|T\zeta\|^2 \right| \leq \varepsilon \left( \|T\eta\|^2 + \frac{(n+1)(1-\delta)}{n(1+\delta)} \|T\zeta\|^2 \right),$$

or equivalently,

$$\sqrt{\frac{(n+1)(1-\delta)(1-\varepsilon)}{n(1+\delta)(1+\varepsilon)}} \|T\zeta\| \leq \|T\eta\| \leq \sqrt{\frac{(n+1)(1-\delta)(1+\varepsilon)}{n(1+\delta)(1-\varepsilon)}} \|T\zeta\|.$$

□

**Corollary 3.2.** *Let  $T : \mathcal{H} \rightarrow \mathcal{K}$  be a  $(\delta, \varepsilon)$ -orthogonality preserving mapping. If  $\eta, \zeta \in \mathcal{H} \setminus \{0\}$  are orthogonal vectors, then*

$$\sqrt{\frac{(1-\delta)(1-\varepsilon)}{(1+\delta)(1+\varepsilon)}} \|T\zeta\| \|\eta\| \leq \|T\eta\| \|\zeta\| \leq \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} \|T\zeta\| \|\eta\|.$$

**Theorem 3.3.** *Let  $T : \mathcal{H} \rightarrow \mathcal{K}$  be a  $(\delta, \varepsilon)$ -orthogonality preserving mapping. Then  $T$  is injective, continuous and satisfies*

$$\frac{1}{\theta} \gamma \|\eta\| \leq \|T\eta\| \leq \theta \gamma \|\eta\|$$

for all  $\eta \in \mathcal{H}$ ,  $\gamma \in [ [T], \|T\| ]$  and  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ .

*Proof.* Let  $\eta, \zeta \in \mathcal{H} \setminus \{0\}$ . Choose  $\eta_1, \eta_2 \in \mathcal{H} \setminus \{0\}$  such that

$$\eta = \eta_1 + \eta_2, \quad \eta_1 \in \{\lambda\zeta : \lambda \in \mathbb{C}\}, \quad |(\eta_1, \eta_2)| = 0 \leq \delta \|\eta_1\| \|\eta_2\|, \quad (3.1)$$

whence

$$\|\eta\|^2 = \|\eta_1\|^2 + \|\eta_2\|^2, \quad \|\eta_1\| \leq \|\eta\|, \quad \|\eta_2\| \leq \|\eta\|. \quad (3.2)$$

By Corollary 3.2, we get

$$\sqrt{\frac{(1-\delta)(1-\varepsilon)}{(1+\delta)(1+\varepsilon)}} \|T\eta_1\| \|\eta_2\| \leq \|T\eta_2\| \|\eta_1\| \leq \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} \|T\eta_1\| \|\eta_2\|. \quad (3.3)$$

So, we reach

$$\begin{aligned}
\|T\eta\|^2 &= \|T\eta_1\|^2 + 2\operatorname{Re}(T\eta_1, T\eta_2) + \|T\eta_2\|^2 \quad (\text{since } \eta = \eta_1 + \eta_2) \\
&\leq \|T\eta_1\|^2 + 2|(T\eta_1, T\eta_2)| + \|T\eta_2\|^2 \\
&\quad \left( \text{since } \frac{\|T\eta_1\|}{\|\eta_1\|} = \frac{\|T\zeta\|}{\|\zeta\|}, |(T\eta_1, T\eta_2)| \leq \delta\|\eta_1\| \|\eta_2\|, \right. \\
&\quad \left. T \text{ is a } (\delta, \varepsilon)\text{-orthogonality preserving mapping and (3.3)} \right) \\
&\leq \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta_1\|^2 + 2\varepsilon\|T\eta_1\| \|T\eta_2\| + \frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)} \frac{\|T\eta_1\|^2}{\|\eta_1\|^2} \|\eta_2\|^2 \\
&\quad \left( \text{by (3.2), (3.3) and since } \frac{\|T\eta_1\|}{\|\eta_1\|} = \frac{\|T\zeta\|}{\|\zeta\|} \right) \\
&\leq \frac{\|T\zeta\|^2}{\|\eta\|^2} (\|\eta\|^2 - \|\eta_2\|^2) + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} \|T\eta_1\|^2 \times \frac{\|\eta_2\|}{\|\eta_1\|} \\
&\quad + \frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)} \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta_2\|^2 \\
&\quad \left( \text{since } \frac{\|T\eta_1\|}{\|\eta_1\|} = \frac{\|T\zeta\|}{\|\zeta\|} \right) \\
&\leq \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta\|^2 + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta_1\| \|\eta_2\| \\
&\quad + \left( \frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)} - 1 \right) \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta_2\|^2 \\
&\quad \left( \text{since } \|\eta_1\| \leq \|\eta\| \text{ and } \|\eta_2\| \leq \|\eta\| \right) \\
&\leq \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta\|^2 \left( 1 + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} + \left( \frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)} - 1 \right) \right) \\
&= \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta\|^2 \left[ \frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)} + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} \right].
\end{aligned}$$

Thus we have  $\|T\eta\|^2 \leq \frac{\|T\zeta\|^2}{\|\zeta\|^2} \|\eta\|^2 \theta^2$  and hence  $\frac{\|T\eta\|}{\|\eta\|} \leq \theta \frac{\|T\zeta\|}{\|\zeta\|}$ . Since  $\eta$  and  $\zeta$  are arbitrary, we change the order to get  $\frac{\|T\zeta\|}{\|\zeta\|} \leq \theta \frac{\|T\eta\|}{\|\eta\|}$  and finally  $\frac{1}{\theta} \frac{\|T\zeta\|}{\|\zeta\|} \leq \frac{\|T\eta\|}{\|\eta\|} \leq \theta \frac{\|T\zeta\|}{\|\zeta\|}$ . Hence  $T$  is continuous and  $\frac{1}{\theta} \|T\| \leq \frac{\|T\eta\|}{\|\eta\|} \leq \theta \|T\|$ .

Now, for all  $\eta \in \mathcal{H}$  and for all  $\gamma \in [T, \|T\|]$ , we reach

$$\frac{1}{\theta} \gamma \|\eta\| \leq \frac{1}{\theta} \|T\| \|\eta\| \leq \|T\eta\| \leq \theta \|T\| \|\eta\| \leq \theta \gamma \|\eta\|.$$

Thus  $T$  is injective and  $\frac{1}{\theta} \gamma \|\eta\| \leq \|T\eta\| \leq \theta \gamma \|\eta\|$ .  $\square$

The following lemma is a consequences of the discussion in the first section.

**Lemma 3.4.** *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{E}$  be inner product  $\mathcal{A}$ -module with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and let  $\mathcal{E}$  be any minimal projection. Then the following statements hold:*

- (i)  *$x, y \in \mathcal{E}_e$  are  $\delta$ -orthogonal in  $\mathcal{E}_e$  if and only if they are  $\delta$ -orthogonal in  $\mathcal{E}$ .*
- (ii) *If  $T : \mathcal{E} \rightarrow \mathcal{F}$  is an  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping, then  $T_e := T|_{\mathcal{E}_e} : \mathcal{E}_e \rightarrow \mathcal{F}_e$  is a linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping.*

*Proof.* (i) Let  $x, y \in \mathcal{E}_e$ . Then

$$\begin{aligned} x \perp^\delta y \text{ in } \mathcal{E}_e &\Leftrightarrow |(x, y)| \leq \delta \|x\|_{\mathcal{E}_e} \|y\|_{\mathcal{E}_e} \Leftrightarrow \|\langle x, y \rangle\| \leq \delta \|x\|_{\mathcal{E}} \|y\|_{\mathcal{E}} \\ &\Leftrightarrow x \perp^\delta y \text{ in } \mathcal{E}. \end{aligned}$$

(ii) Let  $x \perp^\delta y$  in  $\mathcal{E}_e$ . By (i),  $x \perp^\delta y$  in  $\mathcal{E}$ . Since  $T$  is  $(\delta, \varepsilon)$ -orthogonality preserving, hence  $Tx \perp^\varepsilon Ty$  in  $\mathcal{F}$ . So, by (i),  $T_e x \perp^\varepsilon T_e y$  in  $\mathcal{F}_e$ . Thus  $T_e$  is a linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping.  $\square$

A part of the following lemma can be found in [9, Proposition 3.3]. We, however, prove it for the sake of completeness.

**Proposition 3.5.** *Let  $\mathcal{E}, \mathcal{F}$  be inner product  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T : \mathcal{E} \rightarrow \mathcal{F}$  be an  $\mathcal{A}$ -linear mapping. Suppose that  $T_e := T|_{\mathcal{E}_e} : \mathcal{E}_e \rightarrow \mathcal{F}_e$  for some minimal projection  $\mathcal{E}$ , such that  $0 < [T_e] \leq \|T_e\| < \infty$ . Then*

- (i)  $[T] = [T_e]$ .
- (ii)  $\|T\| = \|T_e\|$ .

*Proof.* (i) Let  $e = \zeta \otimes \zeta$ ,  $f = \eta \otimes \eta$  be minimal projections and let  $u = \eta \otimes \zeta$ . We have

$$\begin{aligned} e \langle Tu, Tu \rangle e &= \langle T(ue), T(ue) \rangle = (T(ue), T(ue))e \\ &= \|T(ue)\|_{\mathcal{F}_f}^2 e \geq [T_e]^2 \|ue\|^2 e = [T_e]^2 \left\| (\eta \otimes \zeta)(\zeta \otimes \zeta) \right\|^2 e \\ &= [T_e]^2 \left\| \|\zeta\|^2 \eta \otimes \zeta \right\|^2 e = [T_e]^2 \|u\|^2 e. \end{aligned}$$

Hence

$$[T_e]^2 \|u\|^2 \leq \|e \langle Tu, Tu \rangle e\| \leq \sup\{\|e \langle Tu, Tu \rangle e\| : \|e\| = 1\} = \|Tu\|^2.$$

Hence  $[T_e]\|u\| \leq \|Tu\|$ , which shows  $[T_e] \leq [T]$ . Since  $[T_e] \geq [T]$ , thus we reach  $[T_e] = [T]$ .

(ii) The proof is similar to (i).  $\square$

We are now in a position to establish one of our main results. In fact, in the sequel we provide a version of Theorem 3.3 in the setting of inner product  $C^*$ -modules.

**Theorem 3.6.** *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{E}, \mathcal{F}$  be inner product  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and let  $T : \mathcal{E} \rightarrow \mathcal{F}$  be a nonzero  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping. Then*

(i)  $0 < [T] \leq \|T\| < \infty$ .

(ii)  $\frac{1}{\theta^2} \gamma^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \theta^2 \gamma^2 \langle x, x \rangle$

for all  $x \in \mathcal{E}$ ,  $\gamma \in [ [T], \|T\| ]$  and  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ .

(iii)  $\| \langle Tx, Ty \rangle - \gamma^2 \langle x, y \rangle \| \leq 4(1 - \frac{1}{\theta^2}) \min \{ \gamma^2 \|x\| \|y\|, \|Tx\| \|Ty\| \}$   
for all  $x, y \in \mathcal{E}$  and for all  $\gamma \in [ [T], \|T\| ]$ .

*Proof.* Let  $e = \eta \otimes \eta$  be a minimal projection. From Lemma 3.4 it follows that  $T_e := T|_{\mathcal{E}_e} : \mathcal{E}_e \rightarrow \mathcal{F}_e$  is a linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping. Hence Theorem 3.3 implies  $T_e$  is injective,  $0 < [T_e] \leq \|T_e\| < \infty$  and satisfies

$$\frac{1}{\theta} \gamma \|xe\| \leq \|T_e(xe)\| \leq \theta \gamma \|xe\|, \quad (3.4)$$

for all  $x \in \mathcal{E}$ ,  $\gamma \in [ [T_e], \|T_e\| ]$  and  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}} + 2\varepsilon \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ . Thus by Proposition 3.5,  $0 < [T] \leq \|T\| < \infty$  and it follows from (3.4) that

$$\frac{1}{\theta^2} \gamma^2 (xe, xe) \leq (T_e(xe), T_e(xe)) \leq \theta^2 \gamma^2 (xe, xe),$$

or equivalently,

$$\left( \frac{1}{\theta^2} \gamma^2 \langle x, x \rangle \eta, \eta \right) \leq \left( \langle Tx, Tx \rangle \eta, \eta \right) \leq \left( \theta^2 \gamma^2 \langle x, x \rangle \eta, \eta \right). \quad (3.5)$$

Now (3.5) gives

$$\frac{1}{\theta^2} \gamma^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \theta^2 \gamma^2 \langle x, x \rangle \quad (3.6)$$

for all  $x \in \mathcal{E}$  and for all  $\gamma \in [ [T], \|T\| ]$ . Using the polar identity, we obtain

$$\| \langle Tx, Ty \rangle - \gamma^2 \langle x, y \rangle \| \leq \frac{1}{4} \times 4(1 - \frac{1}{\theta^2}) \gamma^2 (\|x\| + \|y\|)^2 \quad (3.7)$$

$$\leq 2(1 - \frac{1}{\theta^2}) \gamma^2 (\|x\|^2 + \|y\|^2). \quad (3.8)$$

Applying (3.7) for vectors  $\frac{x}{\|x\|}$  and  $\frac{y}{\|y\|}$ , we get

$$\left\| \left\langle T\left(\frac{x}{\|x\|}\right), T\left(\frac{y}{\|y\|}\right) \right\rangle - \gamma^2 \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \right\| \leq 4\left(1 - \frac{1}{\theta^2}\right)\gamma^2,$$

or equivalently,

$$\left\| \langle Tx, Ty \rangle - \gamma^2 \langle x, y \rangle \right\| \leq 4\left(1 - \frac{1}{\theta^2}\right)\gamma^2 \|x\| \|y\|. \quad (3.9)$$

Furthermore (3.6) implies

$$\frac{1}{\theta^2} \frac{1}{\gamma^2} \langle Tx, Tx \rangle \leq \langle x, x \rangle \leq \theta^2 \frac{1}{\gamma^2} \langle Tx, Tx \rangle. \quad (3.10)$$

Similar to (3.9), by (3.10) we reach

$$\left\| \langle Tx, Ty \rangle - \gamma^2 \langle x, y \rangle \right\| \leq 4\left(1 - \frac{1}{\theta^2}\right) \|Tx\| \|Ty\|, \quad (3.11)$$

for all  $x, y \in \mathcal{E}$  and for all  $\gamma \in [[T], \|T\|]$ . Thus, by (3.9) and (3.11), (iii) follows.  $\square$

Next we obtain a sufficient condition for an  $\mathcal{A}$ -linear mapping to be  $(\delta, \varepsilon)$ -orthogonality preserving.

**Corollary 3.7.** *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{E}, \mathcal{F}$  be inner product  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and let  $T : \mathcal{E} \rightarrow \mathcal{F}$  be a nonzero  $\mathcal{A}$ -linear such that*

$$\begin{aligned} \frac{2\delta}{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)} \gamma^2 \langle x, x \rangle &\leq \langle Tx, Tx \rangle \\ &\leq \frac{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)}{2\delta} \gamma^2 \langle x, x \rangle \end{aligned}$$

for all  $x \in \mathcal{E}$  and for some  $\gamma \in [[T], \|T\|]$ . Then  $T$  is  $(\delta, \varepsilon)$ -orthogonality preserving.

*Proof.* Let  $x, y \in \mathcal{E}$  and  $x \perp^\delta y$ . Then  $\|\langle x, y \rangle\| \leq \delta \|x\| \|y\|$ . As in the proof of Theorem 3.6 (iii) we have

$$\left\| \langle Tx, Ty \rangle - \gamma^2 \langle x, y \rangle \right\| \leq 4 \left( 1 - \frac{2\delta}{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)} \right) \|Tx\| \|Ty\|.$$

Hence

$$\begin{aligned}
& \|\langle Tx, Ty \rangle\| \\
& \leq \|\langle Tx, Ty \rangle - \gamma^2 \langle x, y \rangle\| + \gamma^2 \|\langle x, y \rangle\| \\
& \leq 4 \left( 1 - \frac{2\delta}{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)} \right) \|Tx\| \|Ty\| + \gamma^2 \delta \|x\| \|y\| \\
& \leq 4 \left( 1 - \frac{2\delta}{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)} \right) \|Tx\| \|Ty\| \\
& \quad + \gamma^2 \delta \frac{\sqrt{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)}}{\gamma \sqrt{2\delta}} \|Tx\| \\
& \quad \cdot \frac{\sqrt{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)}}{\gamma \sqrt{2\delta}} \|Ty\| \\
& \leq \left[ 4 \left( 1 - \frac{2\delta}{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)} \right) + \frac{\sqrt{(4-\varepsilon)^2 + 16\delta} - (4-\varepsilon)}{2} \right] \\
& \quad \cdot \|Tx\| \|Ty\| = \varepsilon \|Tx\| \|Ty\|.
\end{aligned}$$

Thus  $Tx \perp^\varepsilon Ty$ . □

Let us quote a result from [17].

**Lemma 3.8.** [17, Theorem 3.4] *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{H}, \mathcal{K}$  be Hilbert spaces and let  $T : \mathcal{H} \rightarrow \mathcal{K}$  be a nonzero  $(\delta, \varepsilon)$ -orthogonality preserving mapping. Then  $T$  satisfies  $\theta \|T\| \|\xi\| \leq \|T\xi\|$  for all  $\xi \in \mathcal{H}$ , with  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ .*

**Theorem 3.9.** *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{E}, \mathcal{F}$  be Hilbert  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and let  $T : \mathcal{E} \rightarrow \mathcal{F}$  be a nonzero  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping. Then*

- (i)  $\frac{(1+\delta)(1-\varepsilon)}{(1-\delta)(1+\varepsilon)} \|T\|^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle$   
for all  $x \in \mathcal{E}$ .
- (ii)  $\|\langle Tx, Ty \rangle - \|T\|^2 \langle x, y \rangle\| \leq \frac{4(\varepsilon-\delta)}{(1-\delta)(1+\varepsilon)} \|Tx\| \|Ty\|$   
for all  $x, y \in \mathcal{E}$ .

*Proof.* By Lemma 3.8, we have  $\theta \|T\| \|\xi\| \leq \|T\xi\| \leq \|T\| \|\xi\| \leq \frac{1}{\theta} \|T\| \|\xi\|$  for all  $\xi \in \mathcal{H}$ , with  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ . Thus the proof is similar to the proof of Theorem 3.6 and so we omit it. □

Now, we are going to show some applications of the above theorems, which generalize some results from [4, 9, 17, 19, 20].



As a consequence of Theorem 3.6 and Theorem 3.9, we have the following result.

**Corollary 3.10.** *Let  $0 \leq \varepsilon < \delta < 1$ . Let  $\mathcal{E}, \mathcal{F}$  be Hilbert  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T : \mathcal{E} \rightarrow \mathcal{F}$  be an  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping. Then  $T = 0$ .*

*Proof.* We suppose, for a contradiction, that there is a nonzero  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping with  $0 \leq \varepsilon < \delta < 1$ . According to Theorem 3.6 (i),  $0 < [T] \leq \|T\| < \infty$  and also by Theorem 3.9, we have  $\frac{1}{\theta^2} \|T\|^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle$  for all  $x \in \mathcal{E}$ , with  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ . Since  $\theta < 1$ , we obtain

$$0 < \|T\|^2 \langle x, x \rangle < \frac{1}{\theta^2} \|T\|^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle$$

for all  $x \in \mathcal{E}$ , a contradiction. Therefore,  $T = 0$ .  $\square$

**Corollary 3.11.** *Let  $\delta, \varepsilon \in [0, 1)$ . Let  $\mathcal{E}, \mathcal{F}$  be Hilbert  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and let for any  $n \in \mathbb{N}$ ,  $T_n : \mathcal{E} \rightarrow \mathcal{F}$  be an  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping. If  $T : \mathcal{E} \rightarrow \mathcal{F}$  is a bounded linear mapping such that  $T_n \rightarrow T$ , then  $T$  is  $\varphi$ -orthogonality preserving with  $\varphi = \frac{4(\varepsilon-\delta)}{(1-\delta)(1+\varepsilon)}$ .*

*Proof.* Let  $x, y \in \mathcal{E}$  and  $x \perp y$ . Hence for any  $n \in \mathbb{N}$ , by Theorem 3.9 (ii), we have  $\|\langle T_n x, T_n y \rangle\| \leq \varphi \|T_n x\| \|T_n y\|$ , for all  $x, y \in \mathcal{E}$ , with  $\varphi = \frac{4(\varepsilon-\delta)}{(1-\delta)(1+\varepsilon)}$ . Thus

$$\begin{aligned} \|\langle Tx, Ty \rangle\| &\leq \|\langle Tx, Ty \rangle - \langle T_n x, Ty \rangle\| + \|\langle T_n x, Ty \rangle - \langle T_n x, T_n y \rangle\| \\ &\quad + \|\langle T_n x, T_n y \rangle\| \\ &\leq \|T_n - T\| \|x\| \|Ty\| + \|T_n x\| \|T - T_n\| \|y\| \\ &\quad + \frac{4(\varepsilon - \delta)}{(1 - \delta)(1 + \varepsilon)} \|T_n x\| \|T_n y\|. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain  $\|\langle Tx, Ty \rangle\| \leq \frac{4(\varepsilon-\delta)}{(1-\delta)(1+\varepsilon)} \|Tx\| \|Ty\|$ , which is nothing else but  $Tx \perp^\varphi Ty$ .  $\square$

Taking  $\mathcal{E} = \mathcal{F}$  and  $T = id$ , one obtains, from Theorem 3.9 the following result.

**Corollary 3.12.** *Let  $\delta, \varepsilon, \vartheta \in [0, 1)$ . Let  $\mathcal{E}$  be a Hilbert  $\mathcal{A}$ -module with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and let  $\langle \cdot, \cdot \rangle_1$  and  $\langle \cdot, \cdot \rangle_2$  be two  $\mathcal{A}$ -valued inner products on  $\mathcal{E}$ . If  $\perp_1^\delta \subseteq \perp_2^\varepsilon$ , i.e., if  $\|\langle x, y \rangle_1\| \leq \delta \|x\|_1 \|y\|_1 \Rightarrow \|\langle x, y \rangle_2\| \leq \varepsilon \|x\|_2 \|y\|_2$  for all  $x, y \in \mathcal{E}$ , then there exists  $\gamma > 0$  such that*

- (i)  $\frac{\gamma}{\theta^2} \langle x, x \rangle_1 \leq \langle x, x \rangle_2 \leq \gamma \langle x, x \rangle_1$   
for all  $x \in \mathcal{E}$ , with  $\theta = \sqrt{\frac{(1-\delta)(1+\varepsilon)}{(1+\delta)(1-\varepsilon)}}$ .
- (ii)  $\|\langle x, y \rangle_2 - \gamma \langle x, y \rangle_1\| \leq \varphi \min\{\gamma \|x\|_1 \|y\|_1, \|x\|_2 \|y\|_2\}$   
for all  $x, y \in \mathcal{E}$ , with  $\varphi = \frac{4(\varepsilon-\delta)}{(1-\delta)(1+\varepsilon)}$ .
- (iii)  $\perp_2^\vartheta \subseteq \perp_1^\nu$ , with  $\nu = \vartheta + \frac{4(\varepsilon-\delta)}{(1-\delta)(1+\varepsilon)}$ ,  
which makes sense if  $\nu < 1$ , i.e., for sufficiently small  $\delta, \varepsilon$  and  $\vartheta$ .

Next we obtain some characterizations of the orthogonality preserving mappings in Hilbert  $\mathcal{A}$ -modules.

**Corollary 3.13.** *Let  $\varepsilon \in [0, 1)$ . Let  $\mathcal{E}, \mathcal{F}$  be Hilbert  $\mathcal{A}$ -modules with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$ . For a nonzero  $\mathcal{A}$ -linear mapping  $T : \mathcal{E} \rightarrow \mathcal{F}$  the following statements are equivalent:*

- (i) *There exists  $\gamma > 0$  such that  $\|Tx\| = \gamma\|x\|$  for all  $x \in \mathcal{E}$ .*
- (ii)  *$T$  is injective and  $\frac{\langle Tx, Ty \rangle}{\|Tx\| \|Ty\|} = \frac{\langle x, y \rangle}{\|x\| \|y\|}$  for all  $x, y \in \mathcal{E} \setminus \{0\}$ .*
- (iii)  *$|x| = |y| \Rightarrow |Tx| = |Ty|$  for all  $x, y \in \mathcal{E}$ .*
- (iv)  *$|x| \leq |y| \Rightarrow |Tx| \leq |Ty|$  for all  $x, y \in \mathcal{E}$ .*
- (v)  *$T$  is strongly orthogonality preserving.*
- (vi)  *$T$  is orthogonality preserving.*
- (vii)  *$T$  is strongly  $(\varepsilon, \varepsilon)$ -orthogonality preserving.*
- (viii)  *$T$  is  $(\varepsilon, \varepsilon)$ -orthogonality preserving.*

*Proof.* It follows from Theorem (4.6) and Corollary (4.11) of [19] we have (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v)  $\Leftrightarrow$  (vi).

(ii)  $\Rightarrow$  (vii) and (vii)  $\Rightarrow$  (viii) are trivial.

To prove (viii)  $\Rightarrow$  (i), let  $\delta := \varepsilon$ . From Theorem 3.9 we obtain

$$\frac{(1+\varepsilon)(1-\varepsilon)}{(1-\varepsilon)(1+\varepsilon)} \|T\|^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle$$

for all  $x \in \mathcal{E}$ . Thus  $\langle Tx, Tx \rangle = \|T\|^2 \langle x, x \rangle$  for all  $x \in \mathcal{E}$ .  $\square$

The following example shows that conditions (iii)-(viii) in Corollary 3.13 are not equivalent to conditions (i)-(ii), even in the case  $\varepsilon = 0$ , in an arbitrary Hilbert  $\mathcal{A}$ -module.

**Example 3.14.** Following [19, Example 4.7], let  $\Omega$  be a locally compact Hausdorff space. Let us take  $\mathcal{E} = \mathcal{F} = C_0(\Omega)$ , the  $C^*$ -algebra of all continuous complex-valued functions vanishing at infinity on  $\Omega$ . For a nonzero function

$f_0 \in C_0(\Omega)$ , suppose that  $T : C_0(\Omega) \longrightarrow C_0(\Omega)$  is given by  $T(g) = f_0 g$ . Obviously  $T$  is  $C_0(\Omega)$ -linear and satisfies conditions (iii)-(viii) but need not satisfies conditions (i)-(ii). Indeed, if there exists  $\gamma > 0$  such that  $\|T(g)\| = \gamma\|g\|$  for all  $g \in C_0(\Omega)$ , then  $\frac{1}{\gamma^2} \overline{f_0} f_0 g = g$  for all  $g \in C_0(\Omega)$  and hence,  $\frac{1}{\gamma^2} \overline{f_0} f_0$  is the identity in  $C_0(\Omega)$ , which is a contradiction.

Note that the assumption of  $\mathcal{A}$ -linearity, even in the case  $\varepsilon = 0$  and  $\mathcal{E} = \mathcal{F} = \mathcal{A} = \mathbb{B}(\mathcal{H})$ , is necessary in Corollary 3.13 as one can see from the following example.

**Example 3.15.** Let  $\mathcal{E} = \mathcal{F} = \mathbb{B}(\mathcal{H})$  and let  $P \in \mathbb{B}(\mathcal{H})$  be a nontrivial projection. Then there exists  $S_1 \in \mathbb{B}(\mathcal{H})$  such that  $S_1 P \neq P S_1$ . Hence there exists  $S_2 \in \mathbb{B}(\mathcal{H})$  such that  $S_2(S_1 P - P S_1) \neq 0$ . Now, the mapping  $T : \mathbb{B}(\mathcal{H}) \longrightarrow \mathbb{B}(\mathcal{H})$  defined by  $T(S) = S P$  is orthogonality preserving. Since  $T(S_2 S_1) - T(S_2) S_1 = S_2(S_1 P - P S_1) \neq 0$ , so  $T$  is not  $\mathbb{B}(\mathcal{H})$ -linear. But  $T$  does not satisfy (i). Indeed, if there exists  $\gamma > 0$  such that  $\|T(S)\| = \gamma\|S\|$  for all  $S \in \mathbb{B}(\mathcal{H})$ , then for  $S = P$  we get  $\gamma = 1$ . But  $P$  is a nontrivial projection and we obtain a contradiction; see [9, Example 3.2].

**Corollary 3.16.** Let  $\delta, \varepsilon \in [0, 1)$  and let  $\mathcal{E}, \mathcal{F}$  be Hilbert  $\mathcal{A}$ -modules. The following statements hold:

- (i) If  $S : \mathcal{E} \longrightarrow \mathcal{E}$  is a linear  $(\delta, \delta)$ -orthogonality preserving mapping and  $T$  is  $(\delta, \varepsilon)$ -orthogonality preserving, then  $TS$  is linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping.
- (ii) If  $S : \mathcal{F} \longrightarrow \mathcal{F}$  is a nonzero  $\mathcal{A}$ -linear  $(\varepsilon, \varepsilon)$ -orthogonality preserving mapping with  $\mathbb{K}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$  and  $T$  is an  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving mapping, then  $ST$  is  $\mathcal{A}$ -linear  $(\delta, \varepsilon)$ -orthogonality preserving.

*Proof.* The proof immediately follows from the definition of a  $(\delta, \varepsilon)$ -orthogonality preserving mapping and the equivalence (i) $\Leftrightarrow$ (iv) of Corollary 3.13.  $\square$

**Acknowledgements** The authors would like to thank the referee for several useful comments.

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